

EXTENDED SIMPLICIAL RATIONAL NOMIZU'S THEOREM

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ABSTRACT. For a torsion-free virtually polycyclic group Γ , we give a canonical homomorphism from certain finite-dimensional cochain complex to the \mathbb{Q} -polynomial de Rham complex of the simplicial classifying space $B\Gamma$ which induces a cohomology isomorphism. By this result, we obtain the Sullivan's minimal model of certain differential graded algebra defined on $B\Gamma$ and we obtain new examples of hard Lefschetz symplectic manifolds and hard Lefschetz contact manifolds.

1. INTRODUCTION

A group Γ is polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. For a polycyclic group Γ , we denote $\text{rank } \Gamma = \sum_{i=1}^{i=k} \text{rank } \Gamma_{i-1}/\Gamma_i$. Let Γ be a torsion-free virtually polycyclic group. For a representation of Γ into a \mathbb{Q} -algebraic group G , if the image $\rho(\Gamma)$ is Zariski-dense in G , then we have $\dim U \leq \text{rank } \Gamma$ where U is the unipotent radical of G (see [29, Lemma 4.36.]). We say that $\rho : \Gamma \rightarrow G$ is a full representation if $\dim U = \text{rank } \Gamma$.

For a simplicial complex K with $\pi_1 K = \Gamma$ and a Γ -module V with a finite dimensional \mathbb{Q} -vector space, considering V as a local system on K , we can define the \mathbb{Q} -polynomial de Rham complex $A_p^*(K, V)$ of K with values in local system V . We also consider $V = \varinjlim V_i$ for an inductive system of finite dimensional $\pi_1(K)$ -modules V_i . We define

$$A_p^*(K, V) = \varinjlim A_p^*(K, V_i).$$

Let Γ be a torsion-free virtually polycyclic group. We consider the following situation:

- We have a \mathbb{Q} -algebraic group G and an injective representation $\rho : \Gamma \rightarrow G$ such that the image $\rho(\Gamma)$ is Zariski-dense in G .
- $\rho : \Gamma \rightarrow G$ is a full representation.

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We consider the simplicial classifying space $B\Gamma$ of a torsion-free virtually polycyclic group Γ . For a rational G -module V , we consider the \mathbb{Q} -polynomial de Rham complex $A_p^*(B\Gamma, V)$. We also consider the complex of " G -invariant differential forms" on U . We can take a splitting $G = T \ltimes U$ such that U is the unipotent radical of G and T is a maximal reductive subgroup of G (see [27]). For the Lie algebra \mathfrak{u} of U , we consider the cochain complex $(\bigwedge \mathfrak{u}^* \otimes V)^T$ of the T -invariant elements of the cochain complex of the Lie algebra \mathfrak{u} with values in V . Then, in this paper we show the following result.

Theorem 1.1. *We have an explicit map $(\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow A_p^*(B\Gamma, V)$ which induces a cohomology isomorphism.*

Remark 1.1. It is known that a \mathbb{Q} -algebraic group G and an injective representation $\rho : \Gamma \rightarrow G$ so that

- We have a \mathbb{Q} -algebraic group G and an injective representation $\rho : \Gamma \rightarrow G$ such that the image $\rho(\Gamma)$ is Zariski-dense in G .
- $\rho : \Gamma \rightarrow G$ is a full representation.
- The centralizer $Z_G(U)$ of U is contained in U .

exist and such G is unique up to isomorphism of \mathbb{Q} -algebraic groups ([2, Appendix A.]). Such G is called the algebraic hull of Γ .

For a representation $\phi : \Gamma \rightarrow GL(V)$ with a finite dimensional \mathbb{Q} -vector space V , consider the representation $\rho \times \phi : \Gamma \rightarrow G \times GL(V)$ and take the Zariski-closure G^ϕ of its image. Then the representation $\Gamma \rightarrow G^\phi$ is also a full representation and $\phi : \Gamma \rightarrow GL(V)$ is extended to a rational representation $G^\phi \rightarrow GL(V)$ (see [20, Section 3]). Thus, for any representation $\phi : \Gamma \rightarrow GL(V)$ with a finite dimensional \mathbb{Q} -vector space V , applying Theorem 1.1, the cohomology of the \mathbb{Q} -polynomial de Rham complex $A_p^*(B\Gamma, V)$ is computed by the finite-dimensional cochain complex $(\bigwedge \mathfrak{u}^* \otimes V)^T$.

Theorem 1.1 can be regarded as a generalization of simplicial rational version of Nomizu's Theorem ([28], [23]). Let N be a simply connected nilpotent Lie group and \mathfrak{n} be the Lie algebra of N . We suppose that N has a lattice (i.e. cocompact discrete subgroup) Γ . We consider the nilmanifold $\Gamma \backslash N$. Then, considering the cochain complex $\bigwedge \mathfrak{n}^*$ of the Lie algebra as the space of the invariant differential forms, in [28], Nomizu proves that the canonical inclusion $\bigwedge \mathfrak{n}^* \subset A^*(\Gamma \backslash N)$ induces a cohomology isomorphism

$$H^*(\mathfrak{n}, \mathbb{R}) \cong H^*(\Gamma \backslash N, \mathbb{R})$$

where $A^*(\Gamma \backslash N)$ is the de Rham complex of $\Gamma \backslash N$.

In [23], Lambe and Priddy give a simplicial rational version of Nomizu's theorem. For simply connected nilpotent Lie group N with a lattice Γ , Γ is a torsion-free finitely generated nilpotent group and a nilmanifold $\Gamma \backslash N$ is an aspherical manifold with the fundamental group Γ . Conversely any torsion-free finitely generated nilpotent group Γ can be embedded in a simply connected nilpotent Lie group N whose Lie algebra \mathfrak{n} admits a \mathbb{Q} -structure $\mathfrak{n}_{\mathbb{Q}}$ (see [29]). In [23], considering the simplicial classifying space $B\Gamma$ of a

torsion-free finitely generated nilpotent group Γ and the \mathbb{Q} -polynomial de Rham complex $A_p^*(B\Gamma, \mathbb{Q})$, Lambe and Priddy construct an explicit map $\bigwedge \mathfrak{n}_{\mathbb{Q}} \rightarrow A_p^*(B\Gamma, \mathbb{Q})$ which induces a cohomology isomorphism. A simply connected nilpotent Lie group N can be considered as a real unipotent algebraic group with a \mathbb{Q} -structure $N(\mathbb{Q})$ so that $\Gamma \subset N(\mathbb{Q})$. Regarding a torsion-free finitely generated nilpotent group Γ as a polycyclic group, we can say that the \mathbb{Q} -algebraic group $N(\mathbb{Q})$ is the algebraic hull of Γ (see [29]).

Nomizu's theorem gives an important fact on the theory of Sullivan's minimal model. We can say that the Differential graded algebra (shortly DGA) $\bigwedge \mathfrak{n}^*$ is the minimal model of $A^*(\Gamma \backslash N)$ (see [11]).

We can generalize this fact. For a torsion-free virtually polycyclic group Γ , take G the algebraic hull of Γ and a splitting $G = T \ltimes U$ for a maximal reductive subgroup T . Denote by \mathfrak{u} the Lie algebra of the unipotent hull U of Γ . Consider the coordinate ring $\mathbb{Q}[T]$ of the algebraic group T . By the composition $\Gamma \rightarrow G \rightarrow T$ where $G \rightarrow T$ is the projection, since $\mathbb{Q}[T]$ is a rational T -module, we regard $\mathbb{Q}[T]$ as a Γ -module. Consider the \mathbb{Q} -polynomial de Rham complex $A_p^*(B\Gamma, \mathbb{Q}[T])$. Then $A_p^*(B\Gamma, \mathbb{Q}[T])$ is a differential graded algebra (DGA). By Theorem 1.1, we have the following result.

Theorem 1.2. *We have an explicit DGA map $\bigwedge \mathfrak{u}^* \rightarrow A_p^*(B\Gamma, \mathbb{Q}[T])$ which induces a cohomology isomorphism. Hence $\bigwedge \mathfrak{u}^*$ is the minimal model of $A_p^*(B\Gamma, \mathbb{Q}[T])$.*

See [19] for the similar result on solvmanifolds where a solvmanifold is the compact quotient $\Gamma \backslash S$ of a simply connected solvable Lie group S by a lattice Γ .

An infra-solvmanifold is a manifold of the form G/Δ , where G is a simply connected solvable Lie group, and Δ is a torsion-free subgroup of $\text{Aut}(G) \ltimes G$ such that the closure of $h(\Delta)$ in $\text{Aut}(G)$ is compact where $h : \text{Aut}(G) \ltimes G \rightarrow \text{Aut}(G)$ is the projection. An infra-solvmanifold is a generalization of a solvmanifold. An infra-solvmanifold is a aspherical manifold with a virtually polycyclic group. Theorem 1.1 is useful for finding symplectic structures on infra-solvmanifolds M whose cohomology classes in $H^2(M, \mathbb{Z})$. By finding such symplectic infra-solvmanifolds, we obtain the following results.

Theorem 1.3. *We obtain the following new examples.*

- *Symplectic blow-ups of complex projective spaces along certain embedded infra-solvmanifolds which satisfy the hard Lefschetz properties. (But, it is not clear whether these manifolds admit Kähler metrics.)*
- *Non Sasakian contact manifolds which satisfy the hard Lefschetz properties in the sense of [6], [25].*

2. COHOMOLOGY OF ALGEBRAIC GROUPS

The purpose of this section is to construct an explicit cochain complex homomorphism which induces the Hochschild isomorphism on the rational cohomology of an algebraic group. For this construction, we are inspired by the simplicial construction of Van Est isomorphism on the continuous cohomology of a Lie group (see [30]). Let G be a \mathbb{Q} -algebraic group. A G -module is called rational if it is the sum of finite-dimensional G -stable subspaces $\{V_i\}$ so that each V_i comes from a rational representation. For a rational G -module V , we define the rational cohomology $H^*(G, V) = \text{Ext}_G^*(\mathbb{Q}, V)$ as [15] and [22]. We have the standard resolution. We denote by $C^p(G, V)$ the set of the V -valued rational functions on $\underbrace{G \times \cdots \times G}_{p+1}$ with

the left- G -action. Consider the sequence

$$V \longrightarrow C^0(G, V) \xrightarrow{d} C^1(G, V) \xrightarrow{d} \cdots$$

such that the first map $V \rightarrow C^0(G, V)$ is the embedding as constant functions and $d : C^p(G, V) \rightarrow C^{p+1}(G, V)$ is given by

$$d\phi(g_0, \dots, g_{p+1}) = \sum (-1)^i \phi(g_0, \dots, \hat{g}_i, \dots, g_{p+1})$$

for $\phi \in C^p(G, V)$, $g_0, \dots, g_{p+1} \in G$. Then the rational cohomology $H^*(G, V)$ is the cohomology of the cochain complex $C^*(G, V)^G$. For the unipotent radical U of G and a maximal reductive subgroup T , we have a splitting $G = T \ltimes U$ ([27]). Let \mathfrak{u} be the Lie algebra of U . Hochschild showed that we have an isomorphism

$$H^*(\mathfrak{u}, V)^T \cong H^*(G, V).$$

The purpose of this section is to represent this isomorphism as a cochain complex homomorphism.

Let $A_a^*(U)$ be the algebraic de Rham complex of the algebraic variety U . Consider the coordinate ring $\mathbb{Q}[U]$ of U . By the spectral sequence as [12, Proposition 3.4], we have $H^1(G, \mathbb{Q}[U] \otimes W) = H^1(U, \mathbb{Q}[U] \otimes W)^T = 0$ for any finite dimensional rational G -module W . This implies that $\mathbb{Q}[U]$ is an injective G -module (see [16]). We have $A_a^*(U) \cong \text{Hom}_{\mathbb{Q}}(\bigwedge \mathfrak{u}, \mathbb{Q}[U])$ and hence $A_p^*(U) \otimes V$ is an injective G -module (see [22]). Since the exponential map $\exp : \mathfrak{u} \rightarrow U$ is an isomorphism of \mathbb{Q} -algebraic variety, we have $H^0(A_a^*(U) \otimes V) = V$ and $H^*(A_a^*(U) \otimes V) = 0$ for $* > 0$. Hence the sequence

$$V \longrightarrow A_a^0(U) \otimes V \xrightarrow{d} A_a^1(U) \otimes V \xrightarrow{d} \cdots$$

is an injective resolution.

By a splitting $G = T \ltimes U$, we have the homomorphism $\alpha : G \rightarrow \text{Aut}(U) \ltimes U$. We define the map

$$\sigma^p(\cdot)(\cdot) : \underbrace{G \times \cdots \times G}_{p+1} \times \mathbb{Q}^p \rightarrow U$$

such that $\sigma^0(g_0)(0) = \alpha(g_0)e$, $\sigma^1(g_0, g_1)(t_1) = \alpha(g_0)\exp((1-t_1)\log\alpha(g_0^{-1}g_1)e$ and inductively

$$\sigma^p(g_0, \dots, g_p)(t_1, \dots, t_p) = \alpha(g_0)\exp((1-t_1)\log\sigma^{p-1}(g_0^{-1}g_1, \dots, g_0^{-1}g_p)(t_2, \dots, t_p)).$$

It is known that the exponential map $\exp : \mathfrak{u} \rightarrow U$ is an isomorphism of \mathbb{Q} -algebraic variety and $\log : U \rightarrow \mathfrak{u}$ is the inverse. Hence the map

$$\sigma^p : \underbrace{G \times \dots \times G}_{p+1} \times \mathbb{Q}^p \rightarrow U$$

is a homomorphism of \mathbb{Q} -algebraic variety such that for any (t_1, \dots, t_p) , the map

$$\sigma^p(\cdot)(t_1, \dots, t_p) : \underbrace{G \times \dots \times G}_{p+1} \rightarrow U$$

is a G -equivariant map. We note

$$\begin{aligned} \sigma^p(g_0, \dots, g_p)(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{p-1}) \\ = \sigma^{p-1}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_p)(t_1, \dots, t_{p-1}). \end{aligned}$$

For (g_0, \dots, g_p) and $\omega \in A_a^*(U) \otimes V$, considering the map $\sigma^p(g_0, \dots, g_p)(\cdot) : \mathbb{Q}^p \rightarrow U \cong \mathfrak{u}$ which is a homomorphism of \mathbb{Q} -algebraic variety, we have the \mathbb{Q} -algebraic differential form $\sigma^p(g_0, \dots, g_p)^*\omega$ for parameters $(t_1, \dots, t_p) \in \mathbb{Q}^p$. We regard $\sigma^p(g_0, \dots, g_p)^*\omega$ as a \mathbb{Q} -polynomial differential form on \mathbb{R}^p . We define the map $\theta : A_a^*(U) \otimes V \rightarrow C^p(G, V)$ such that

$$\theta(\omega)(g_0, \dots, g_p) = \int_{\Delta} \otimes \text{id}_V \sigma^p(g_0, \dots, g_p)^*\omega$$

where

$$\Delta = \{(1 - t_1 - \dots - t_p, t_1, \dots, t_p) | 0 \leq t_i \leq 1\}$$

is the standard p -simplex for the parameters (t_1, \dots, t_p) . Then we can easily show that the map $\theta : A_a^*(U) \otimes V \rightarrow C^*(G, V)$ is G -equivariant cochain complex homomorphism by Stokes' theorem. (cf. [30, Section 3])

Now we have $(A_a^*(U) \otimes V)^G = (\bigwedge \mathfrak{u}^* \otimes V)^T$ where $\bigwedge \mathfrak{u}^* \otimes V$ is the cochain complex of the Lie algebra \mathfrak{u} with values in the \mathfrak{u} -module V . Consider the restriction $\theta : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow C^*(G, V)^G$. Then the induced map $\theta : H^*(\mathfrak{u}, V)^T \rightarrow H^*(G, V)$ is identified with the map $\text{Ext}_{\mathcal{G}}^*(\mathbb{Q}, V) \rightarrow \text{Ext}_{\mathcal{G}}^*(\mathbb{Q}, V)$ induced by the identity map $V \rightarrow V$. Hence we have the following result.

Theorem 2.1. *The map $\theta : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow C^*(G, V)^G$ induces a cohomology isomorphism*

$$H^*(\mathfrak{u}, V)^T \cong H^*(G, V).$$

3. SIMPLICIAL DE RHAM THEORY

We denote by $A_p^*(n)$ the \mathbb{Q} -DGA which is generated by t_0, \dots, t_n of degree 0 and dt_0, \dots, dt_n of degree 1 with the relations $t_0 + \dots + t_n = 1$ and $dt_0 + \dots + dt_n = 0$. We can regard $A_p^*(n)$ as the \mathbb{Q} -polynomial de Rham complex on the standard n -simplex Δ^n . We define the map $\int_{\Delta^n} : A_p^*(n) \rightarrow \mathbb{Q}$ by the ordinary Riemannian integral. Let K be a simplicial complex with a universal covering complex \tilde{K} . We denote by $n(\sigma)$ the dimension of a simplex $\sigma \in K$. Let V be a finite-dimensional \mathbb{Q} -vector space which is a $\pi_1(K)$ -module. We denote by $A_p^*(K, V)$ the space of collections $\{\omega_\sigma \in A_p^*(n(\sigma)) \otimes V\}_{\sigma \in \tilde{K}}$ such that:

- $\{\omega_\sigma\}_{\sigma \in \tilde{K}}$ are compatible under restrictions to faces i.e. $i^* \omega_\sigma = \omega_\tau$ for the inclusion $i : \tau \rightarrow \sigma$ of a face.
- $\{\omega_\sigma\}_{\sigma \in \tilde{K}}$ are invariant under the $\pi_1(K)$ -action i.e. $\gamma \cdot \omega_{\gamma\sigma} = \omega_\sigma$ for any $\gamma \in \pi_1(K)$.

We call $A_p^*(K, V)$ the \mathbb{Q} -polynomial de Rham complex of K with values in local system V . The space $A_p^*(K, V)$ with the exterior derivation is a cochain complex. Let $C^*(K, V) = (C^*(\tilde{K}) \otimes V)^\Gamma$ be the cochain complex of simplicial cochains with values in the local system V . Define the map $\iota : A_p^*(K, V) \rightarrow C^n(K, V)$ such that for $\sigma \in K$ with $n(\sigma) = n$

$$\iota(\{\omega\})(\sigma) = \int_{\Delta^n} \otimes \text{id}_V(\omega_\sigma).$$

Then, this map is a cochain complex homomorphism and this map induces a cohomology isomorphism (see [8, Chapter 9], [13, Chapter 12-14]).

Let $V = \varinjlim V_i$ for an inductive system of finite dimensional $\pi_1(K)$ -modules. We define

$$A_p^*(K, V) = \varinjlim A_p^*(K, V_i)$$

Example 1. (cf.[10]) Let T be a reductive \mathbb{Q} -algebraic group and $\rho : \pi_1(K) \rightarrow T$ a representation. Consider the coordinate ring $\mathbb{Q}[T]$ of T . Then as a (T, T) -bimodule, we have

$$\mathbb{Q}[T] = \bigoplus V_\alpha^* \otimes V_\alpha$$

such that $\{V_\alpha\}$ is a set of isomorphism classes of irreducible right T -module ([10, Proposition 3.1]). We regard $\mathbb{Q}[T]$ as a $\pi_1(K)$ -module by ρ . Then we have

$$A_p^*(K, \mathbb{Q}[T]) = \bigoplus A_p^*(K, V_\alpha^*) \otimes V_\alpha$$

and it is a DGA.

Let Γ be a discrete group. For a Γ -module V , we define the group cohomology $H^*(\Gamma, V) = \text{Ext}_\Gamma^*(\mathbb{Q}, V)$. We have the standard resolution. We denote by $C^p(\Gamma, V)$ the set of the V -valued functions on $\underbrace{\Gamma \times \dots \times \Gamma}_{p+1}$ with

the left- Γ -action. Consider the sequence

$$V \longrightarrow C^0(\Gamma, V) \xrightarrow{d} C^1(\Gamma, V) \xrightarrow{d} \dots$$

such that the first map $V \rightarrow C^0(\Gamma, V)$ is the embedding as constant functions and $d : C^p(\Gamma, V) \rightarrow C^{p+1}(\Gamma, V)$ is given by

$$d\phi(\gamma_0, \dots, \gamma_{p+1}) = \sum (-1)^i \phi(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{p+1})$$

for $\phi \in C^p(\Gamma, V)$, $\gamma_0, \dots, \gamma_{p+1} \in \Gamma$. Then the group cohomology $H^*(\Gamma, V)$ is the cohomology of the cochain complex $C^*(\Gamma, V)^\Gamma$. A representation $\rho : \Gamma \rightarrow G$ induces a homomorphism $\rho^* : H^*(G, V) \rightarrow H^*(\Gamma, V)$. By the map $\theta : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow C^*(G, V)^G$ as Section 2, we give a geometric representation of $\rho^* : H^*(G, V) \rightarrow H^*(\Gamma, V)$.

We define the acyclic simplicial complex $E\Gamma$ with the free discontinuous Γ -action so that:

- For integers $n \geq 0$, simplices of $E\Gamma$ are standard n -simplices $\Delta_{(\gamma_0, \dots, \gamma_n)}$ indexed by Γ^{n+1} .
-

$$(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)_{(\gamma_0, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n)} \in \Delta_{(\gamma_0, \dots, \gamma_{i-1}, \gamma_{i+1}, \dots, \gamma_n)}$$

is identified with

$$(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)_{(\gamma_0, \dots, \gamma_n)} \in \Delta_{(\gamma_0, \dots, \gamma_n)}.$$

- For $\gamma \in \Gamma$, the action is given by

$$\gamma \cdot (t_1, \dots, t_n)_{(\gamma_0, \dots, \gamma_n)} = (t_1, \dots, t_n)_{(\gamma\gamma_0, \dots, \gamma\gamma_n)}.$$

We define $B\Gamma$ the quotient of $E\Gamma$ by the Γ -action. Then the simplicial complex $B\Gamma$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. For a finite dimensional Γ -module V , we consider the \mathbb{Q} -polynomial de Rham complex $A_p^*(B\Gamma, V)$ of $B\Gamma$ with values in local system V . Define the map $\iota : A_p^n(B\Gamma, V) \rightarrow C^n(\Gamma, V)^\Gamma$ as

$$\iota(\{\omega_\sigma\}_{\sigma \in B\Gamma})(\gamma_0, \dots, \gamma_n) = \int_{\Delta_{(\gamma_0, \dots, \gamma_n)}} \omega_{\Delta_{(\gamma_0, \dots, \gamma_n)}}.$$

Since we can identify $C^*(\Gamma, V)^\Gamma$ with the cochain complex $C^*(B\Gamma, V)$, the map $\iota : A_p^n(B\Gamma, V) \rightarrow C^p(\Gamma, V)^\Gamma$ induces a cohomology isomorphism

$$H^*(A_p^*(B\Gamma, V)) \cong H^*(\Gamma, V).$$

We define the map $\psi : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow A_p^*(B\Gamma, V)$ such that

$$\psi(\omega) = \{\sigma^p(\rho(\gamma_0), \dots, \rho(\gamma_p))^* \omega\}_{\Delta_{(\gamma_0, \dots, \gamma_n)}}$$

where $\sigma^p(\rho(\gamma_0), \dots, \rho(\gamma_p))^* \omega$ is defined in Section 2. By the G -invariance of $\omega \in (\bigwedge \mathfrak{u}^* \otimes V)^T$ and the relation

$$\begin{aligned} \sigma^p(g_0, \dots, g_p)(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{p-1}) \\ = \sigma^{p-1}(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_p)(t_1, \dots, t_{p-1}), \end{aligned}$$

we actually have $\psi(\omega) \in A_p^*(B\Gamma, V)$. Then the map $\psi : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow A_p^*(B\Gamma, V)$ is a cochain complex homomorphism.

We have the commutative diagram

$$\begin{array}{ccc} (\bigwedge \mathfrak{u}^* \otimes V)^T & \xrightarrow{\theta} & C^*(G, V)^G \\ \downarrow \psi & & \downarrow \rho^* \\ A_p^*(B\Gamma, V) & \xrightarrow{\iota} & C^p(\Gamma, V)^\Gamma. \end{array}$$

Hence we have:

Corollary 3.1. *The induced map $\psi^* : H^*(\mathfrak{u}, V)^T \rightarrow H^*(A_p^*(B\Gamma, V))$ is identified with the map $\rho^* : H^*(G, V) \rightarrow H^*(\Gamma, V)$.*

Suppose $V = \mathbb{Q}[T]$. Then we have $(\bigwedge \mathfrak{u}^* \otimes \mathbb{Q}[T])^T = \bigwedge \mathfrak{u}^*$ and we have the DGA map $\psi : \bigwedge \mathfrak{u}^* \rightarrow A_p^*(B\Gamma, \mathbb{Q}[T])$. As we notice in Example 1, in this case Corollary 3.1 is written as the following statement.

Corollary 3.2. *The induced map $\psi^* : H^*(\mathfrak{u}, \mathbb{Q}) \rightarrow H^*(A_p^*(B\Gamma, \mathbb{Q}[T]))$ is identified with the map*

$$\rho^* : \bigoplus H^*(G, V_\alpha^*) \otimes V_\alpha \rightarrow \bigoplus H^*(\Gamma, V_\alpha^*) \otimes V_\alpha.$$

4. CLASSIFYING SPACES OF TORSION-FREE VIRTUALLY POLYCYCLIC GROUPS

Let Γ be a torsion-free virtually polycyclic group, G a \mathbb{Q} -algebraic group and $\rho : \Gamma \rightarrow G$ a representation with the Zariski-dense image. It is known that we have $\dim U \leq \text{rank } \Gamma$ where U is the unipotent radical of G . We say that $\rho : \Gamma \rightarrow G$ is a full representation if $\dim U = \text{rank } \Gamma$.

Theorem 4.1 ([20]). *If $\rho : \Gamma \rightarrow G$ is an injective full representation, then for any rational G -module V the induced map $\rho^* : H^*(G, V) \rightarrow H^*(\Gamma, V)$ is an isomorphism.*

We suppose that $\rho : \Gamma \rightarrow G$ is an injective full representation. Denote by \mathfrak{u} the Lie algebra of the unipotent radical U of G . Take a splitting $G = T \ltimes U$ for a maximal reductive subgroup T . Then by Corollary 3.1 and Theorem 4.1, we have the following fact.

Theorem 4.2. *The map $\psi : (\bigwedge \mathfrak{u}^* \otimes V)^T \rightarrow A_p^*(B\Gamma, V)$ induces a cohomology isomorphism.*

Suppose $V = \mathbb{Q}[T]$ as a Γ -module. By Corollary 3.2, we obtain the following result.

Theorem 4.3. *We have an explicit DGA map $\bigwedge \mathfrak{u}^* \rightarrow A_p^*(B\Gamma, \mathbb{Q}[T])$ which induces a cohomology isomorphism. Hence $\bigwedge \mathfrak{u}^*$ is the minimal model of $A_p^*(B\Gamma, \mathbb{Q}[T])$.*

5. EXAMPLES AND APPLICATIONS TO SYMPLECTIC AND CONTACT GEOMETRY

An infra-solvmanifold is a manifold of the form G/Δ , where G is a simply connected solvable Lie group, and Δ is a torsion-free subgroup of $\text{Aut}(G) \ltimes G$ such that the closure of $h(\Delta)$ in $\text{Aut}(G)$ is compact where $h : \text{Aut}(G) \ltimes G \rightarrow \text{Aut}(G)$ is the projection. An infra-solvmanifold is a generalization of a solvmanifold. An infra-solvmanifold is a aspherical manifold with a virtually polycyclic group.

Let Γ be a torsion-free virtually polycyclic group. Then there exists a compact infra-solvmanifold M with the fundamental group Γ (see [2]). It is known that every compact infra-solvmanifold is smoothly rigid ([2, Corollary 1.5]). Hence we have the canonical correspondence between torsion-free virtually polycyclic groups Γ and infra-solvmanifolds M_Γ with $\pi_1(M_\Gamma) = \Gamma$. Hence by Theorem 4.2, we can study the cohomology of a infra-solvmanifolds M_Γ .

For a torsion-free virtually polycyclic group Γ and a full representation $\rho : \Gamma \rightarrow G$, take a splitting $G = T \ltimes U_\Gamma$ for a maximal reductive subgroup T . Denote by \mathfrak{u} the Lie algebra of the unipotent radical U of G . Then by Theorem 4.2, the cohomology of the DGA $(\bigwedge \mathfrak{u}^*)^T$ is isomorphic to $H^*(M_\Gamma, \mathbb{Q})$. By this isomorphism, we can find an integral symplectic form on the compact infra-solvmanifold M_Γ .

Proposition 5.1. *If there exists a 2-form $\omega \in (\bigwedge^2 \mathfrak{u}^*)^T$ such that $d\omega = 0$ and ω is non-degenerate, then the compact infra-solvmanifold M_Γ admits a integral symplectic form.*

Proof. We suppose that $\text{rank } \Gamma = 2n$. It is sufficient to show that M_Γ admits a symplectic form α such that $[\alpha] \in H^2(M_\Gamma, \mathbb{Q})$. It is known that if we have $a \in H^2(M_\Gamma, \mathbb{R})$ so that $a^n \neq 0$, then we have a symplectic form α which is a representative of a (see [17]).

Now we suppose that there exists a 2-form $\omega \in (\bigwedge^2 \mathfrak{u}^*)^T$ such that $d\omega = 0$ and ω is non-degenerate. Consider the cohomology class $[\omega] \in H^2(M_\Gamma, \mathbb{Q})$. Then, since $\omega \in (\bigwedge^2 \mathfrak{u}^*)^T$ is non-degenerate, we have $[\omega]^n \neq 0$. Hence, taking a symplectic form which is a representative of $[\omega]$, we can prove the proposition. \square

Example 2. Let A be a semi-simple matrix such that $A \in SL_m(\mathbb{Z})$ and for any non-zero integer n we suppose $A^n \neq 1$ where 1 is the unit matrix.

Consider the semi-direct product $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^m$ such that the action of \mathbb{Z} on \mathbb{Z}^m is given by $\mathbb{Z} \ni t \mapsto A^t \in \text{Aut}(\mathbb{Z}^m)$. The group Γ is torsion-free polycyclic of rank $m + 1$. Take $B = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in SL_{m+1}(\mathbb{Z})$. Let T be the Zariski-closure of $\langle B \rangle$ in $SL_{m+1}(\mathbb{Q})$. By the assumption, T is diagonalizable. consider the \mathbb{Q} -algebraic group $G = T \ltimes \mathbb{Q}^{m+1}$ with the unipotent radical \mathbb{Q}^{m+1} . We have the injective homomorphism

$$\rho : \Gamma \ni (t, v) \mapsto (B^t, t, v) \in G$$

where $t \in \mathbb{Z}$ and $v \in \mathbb{Z}^m$. Then we can easily check that $\rho(\Gamma)$ is Zariski-dense in G and so ρ is full. Thus by Theorem 4.2, we have an isomorphism

$$H^*(M_\Gamma, \mathbb{Q}) \cong \left(\bigwedge \mathbb{Q}^{m+1} \right)^T = \left(\bigwedge \mathbb{Q}^{m+1} \right)^{\langle B \rangle}.$$

Additionally, we assume that

$$A = \begin{pmatrix} 1 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

such that $A_1, \dots, A_k \in SL_2(\mathbb{Z})$ and they are semi-simple. By the above assumption, some A_i satisfies $A_i^n \neq 1$ for any non-zero integer n . Then, taking the standard basis $e_1 \dots e_{2k+2}$ of \mathbb{Q}^{2k+2} , we obtain the non-degenerate two form $\omega \in \left(\bigwedge^2 \mathbb{Q}^{2k+2} \right)^{\langle B \rangle}$ such that

$$\omega = e_1 \wedge e_2 + \dots + e_{2k+1} \wedge e_{2k+2}.$$

Hence in this case M_Γ admits a integral symplectic form ω . By [18, Proposition 1.1, Proposition 1.4], we can say that M_Γ is formal in the sense of Sullivan ([7], [31]) and (M_Γ, ω) satisfies the hard Lefschetz property. On the other hand, since the matrix A does not have a finite period, Γ is not virtually nilpotent and hence by the result in [1] M_Γ does not admit a Kähler structure. Using integral symplectic form ω , we construct the following examples:

New Examples 1. Since M_Γ is $2k+2$ -dimensional manifold with integral symplectic form ω , we have a symplectic emmbedding $M_\Gamma \rightarrow \mathbb{C}P^N$ for $N \geq 2k+3$ ([9], [32]). Hence, as McDuff's construction [26], we obtain the symplectic blow-up X_Γ of $\mathbb{C}P^N$ along M_Γ . By [5, Theorem 2.2], the symplectic blow-up X_Γ satisfies the hard Lefschetz property. Moreover, in [24], it is announced that the blow-up along a manifold M symplectically embedded in a large enough complex projective space is formal if and only if M is formal. If this fact true, then X_Γ is formal and hence we obtain a formal and hard Lefschetz symplectic manifolds such that we do not know whether they admit Kähler structures.

New Examples 2. By the integral symplectic form ω on M_Γ , we obtain the principal circle bundle $P_\Gamma \rightarrow M_\Gamma$ associated with $[\omega] \in H^2(M_\Gamma, \mathbb{Z})$. It is known that P_Γ is a regular contact manifold. Recently, the hard Lefschetz property on contact manifolds is defined. (see [6], [25]) In [25], it is proved that the contact manifold P_Γ satisfies hard Lefschetz property if and only if the symplectic manifold M_Γ satisfies hard Lefschetz property. Hence P_Γ satisfies hard Lefschetz property. In [21], it is proved that polycyclic fundamental groups of compact Sasakian manifolds are virtually nilpotent. Since Γ is not virtually nilpotent, the fundamental group of P_Γ is not virtually nilpotent. Hence P_Γ does not admit a Sasakian structure.

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